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A remainder formula of numerical differentiation for the generalized Lagrange interpolation

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ABSTRACT

Through exploiting the generalized Lagrange interpolation for continuous piecewise smooth functions, a remainder term of numerical differentiation is given. Also, the error estimates for the numerical differentiation formula are presented.

1. Introduction

Lagrange interpolation is a simple method for finding the unique polynomial of degree n that exactly passes through $n+1$ points. Let $f : [a, b] \to \mathbb{R}$ be a continuous function in the finite interval $[a, b]$ and $n + 1$ times differentiable. Let $x_0, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$ and let $P_n(x)$ be the associated Lagrange interpolating polynomial of degree n. Then for each $x \in (a, b)$ there exists a point ξ_x such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0) \cdots (x - x_n),$$

where

$$\min\{x, x_0, \ldots, x_n\} < \xi_x < \max\{x, x_0, \ldots, x_n\}.$$

Many textbooks on numerical analysis have provided the error formula. It is obvious that the above equation is invalid for non-smooth functions. However, a new method [1] has been developed by presenting a class of interpolating functions verifying certain approximation error bounds for continuous piecewise smooth functions. Amat et al. defined the generalized divided differences by induction. Applying the generalized Rolle's theorem a generalized Lagrange interpolation formula is obtained.

Numerical differentiation is very important in scientific computing and practical applications. It is mainly used to compute the derivatives of a function at specified points. Numerical differentiation formulas find their applications in the numerical solutions of ordinary differential equations and partial differential equations. A variety of techniques have been developed to construct useful difference formulas for numerical derivatives. Khan et al. [2–7] have presented the explicit forward, backward, and central difference formulas of finite difference approximations for first and higher derivatives based on Taylor series. Numerical differentiation formulas based on Lagrange and Hermite interpolating polynomials may be found in many literatures [8,9]. In a recent paper [10], by means of the generalized Vandermonde determinant, Li constructed an explicit numerical differentiation formula with arbitrary order accuracy for approximating first and higher derivatives which is applicable to unequally or equally spaced data. More recently, in [11] the critical theoretical problems on local Lagrangian

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numerical differentiation have been studied: explicit formulas, local estimate for the remainder, and the highest order of approximation in the case that values of the function at these interpolation nodes have perturbations.

The purpose of this paper is to use the technique in [8] to obtain a remainder of numerical differentiation formula for the Lagrange interpolation in an ECT system of functions by generalizing the definition of divided differences in [1].

2. Main results

We follow the terminology and notations from [1]. Given a collection of increasing continuous functions

$$\{\gamma_i : [a,b] \subset \mathbb{R} \to \mathbb{R}\}_{i=0}^{r},$$

such that $\gamma_i'(x) > 0$ for all $x \in (a,b) \setminus \{y_i\}$, and $\gamma_i'(y_i) = +\infty$, we let the space of continuous

$$\mathcal{F}_{\gamma_i}[a,b] := \left\{ f \in \mathcal{C}[a,b] \mid D_{\gamma_i} f(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{\gamma_i(x+h)-\gamma_i(x)} \right\}.$$

Define $D^0_{\gamma_i} f = f$, $f^m_{\gamma_i} := D^m_{\gamma_i} f = D_{\gamma_i}(D^{m-1}_{\gamma_i} f)$ and a function $f \in \mathcal{F}^m_{\gamma_i}[a,b]$ if and only if there exists $f^{m-1}_{\gamma_i}(x) \in \mathcal{F}_{\gamma_i}[a,b]$, $m \in \mathbb{N}$. For $0 \le p \le j,\ 0 \le s$, Amat [1] defined the following generalized divided differences.

$$f_{\gamma_i}[x_{j-1}, x_j] := \frac{f(x_j) - f(x_{j-1})}{\gamma_i(x_j) - \gamma_i(x_{j-1})},$$

$$f_{\gamma_i}[x_{j-p}, \ldots, x_{j-p+s}] := \frac{f_{\gamma_i}[x_{j-p+1}, \ldots, x_{j-p+s}] - f_{\gamma_i}[x_{j-p}, \ldots, x_{j-p+s-1}]}{\gamma_i(x_{j-p+s}) - \gamma_i(x_{j-p})}.$$

According to the definition of the generalized divided differences and the generalization of Rolle's theorem [1], a generalized Lagrange interpolation is described as follows.

Proposition 1. *Let $f \in \mathcal{F}^{n+1}_{\gamma_i}[a,b]$ and let $x_0, x_1, \ldots, x_n$ be different points in $[a,b]$. Then for $x \in (a,b)$,*

$$f(x) - p_n(x,\gamma_i) = \frac{f^{(n+1)}_{\gamma_i}(\xi)}{(n+1)!}(\gamma_i(x) - \gamma_i(x_0)) \cdots (\gamma_i(x) - \gamma_i(x_n)), \tag{2.1}$$

where

$$p_n(x,\gamma_i) = f(x_0) + \sum_{\nu=1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu](\gamma_i(x) - \gamma_i(x_0)) \cdots (\gamma_i(x) - \gamma_i(x_{\nu-1}))$$

and

$$\xi \in (\min\{x, x_0, \ldots, x_n\}, \max\{x, x_0, \ldots, x_n\}).$$

Actually, $\{(\gamma_i(x))^s\}_{s=1}^n$ are linearly independent functions in $C[a,b]$. Consequently, $p_n(x,\gamma_i)$ can be considered as an interpolating polynomial with a basis $\{(\gamma_i(x))^s\}_{s=1}^n$. Moreover, the definition of the generalized divided differences directly leads to the Newton form of the Lagrange interpolation

$$f(x) = p_n(x,\gamma_i) + f_{\gamma_i}[x, x_0, \ldots, x_n](\gamma_i(x) - \gamma_i(x_0)) \cdots (\gamma_i(x) - \gamma_i(x_n)).$$

Comparing the last equation with (2.1), one readily obtains

$$f_{\gamma_i}[x, x_0, \ldots, x_n] = \frac{f^{(n+1)}_{\gamma_i}(\xi)}{(n+1)!}, \quad x \notin \{x_0, \ldots, x_n\}.$$

All of the above facts are considered if the interpolation points are distinct. Recall that divided differences with repeated points, then as the distinct points coalesce, we also can give a similar definition of generalized divided differences over a set of points with repetitions.

Definition 1. *Let $x_0 \le x_1 \ldots \le x_n$. For $n \ge 1$ we define*

$$f_{\gamma_i}[x_0, \ldots, x_n] = \begin{cases} \dfrac{f_{\gamma_i}[x_1, \ldots, x_n] - f_{\gamma_i}[x_0, \ldots, x_{n-1}]}{\gamma_i(x_n) - \gamma_i(x_0)}, & x_n \neq x_0, \\[2ex] \dfrac{f^{(n)}_{\gamma_i}(x_0)}{n!}, \quad x_n = x_0, \end{cases}$$

where

$$f_{\gamma_i}[x_0] = f(x_0).$$

For convenience, we let $R_n(x,\gamma_i) = f(x) - p_n(x,\gamma_i)$ and let

$$\omega_0(x,\gamma_i) = 1,$$

$$\omega_\nu(x,\gamma_i) = (\gamma_i(x) - \gamma_i(x_0)) \cdots (\gamma_i(x) - \gamma_i(x_{\nu-1})), \quad \nu = 1, 2 \ldots, n.$$

Now, we state our main results.

Theorem 1. *Let $f \in \mathcal{F}^k_{\gamma_i}[a, b]$ and let $x_0, x_1, \ldots, x_n$ be different points in $[a, b]$. For $0 \le k \le m \le n$, then we have*

$$D^k_{\gamma_i} R_n(x, \gamma_i) = k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{m-\nu}] \frac{D^{k-\nu}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i)}{(k-\nu)!} \times (\gamma_i(x) - \gamma_i(x_{m-\nu})) - \sum_{\nu=m+1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i). \tag{2.2}$$

Proof. We shall proceed by induction on m, and we begin with the case $m = k$. First note that

$$\begin{aligned} D^k_{\gamma_i} R_n(x, \gamma_i) &= D^k_{\gamma_i} f(x) - D^k_{\gamma_i} p_n(x, \gamma_i) = k! f_{\gamma_i}[\underbrace{x, \ldots, x}_{k+1}] - \sum_{\nu=k}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i) \\ &= k! \Bigg\{ f_{\gamma_i}[\underbrace{x, \ldots, x}_{k+1}] - f_{\gamma_i}[\underbrace{x, \ldots, x}_{k}, x_0] + \sum_{\nu=0}^{k-1} (f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{k-\nu-1}] - f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu}, x_0, \ldots, x_{k-\nu}]) \\ &\quad + f_{\gamma_i}[x_0, \ldots, x_k] \Bigg\} - \sum_{\nu=k}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i). \end{aligned}$$

By the recursive definition of the generalized divided differences, it is obviously true that

$$D^k_{\gamma_i} R_n(x, \gamma_i) = k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{k-\nu}](\gamma_i(x) - \gamma_i(x_{k-\nu})) - \sum_{\nu=k+1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i).$$

This is in accordance with (2.2). Now let us assume the correctness of Eq. (2.2) for $k \le m < n$ and prove it for $m + 1$. It is clear that

$$\omega_{m+1-\nu}(x, \gamma_i) = \omega_{m-\nu}(x, \gamma_i)(\gamma_i(x) - \gamma_i(x_{m-\nu})).$$

By the Leibnitz formula we have

$$D^{k-\nu}_{\gamma_i} \omega_{m+1-\nu}(x, \gamma_i) = D^{k-\nu}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i)(\gamma_i(x) - \gamma_i(x_{m-\nu})) + (k-\nu) D^{k-\nu-1}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i).$$

This implies that

$$\frac{D^{k-\nu}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i)}{(k-\nu)!}(\gamma_i(x) - \gamma_i(x_{m-\nu})) = \frac{D^{k-\nu}_{\gamma_i} \omega_{m+1-\nu}(x, \gamma_i)}{(k-\nu)!} - \frac{D^{k-\nu-1}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i)}{(k-\nu-1)!}. \tag{2.3}$$

Hence, by Eq. (2.3), we have

$$\begin{aligned} D^k_{\gamma_i} R_n(x, \gamma_i) &= k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{m-\nu}] \left\{ \frac{D^{k-\nu}_{\gamma_i} \omega_{m+1-\nu}(x, \gamma_i)}{(k-\nu)!} - \frac{D^{k-\nu-1}_{\gamma_i} \omega_{m-\nu}(x, \gamma_i)}{(k-\nu-1)!} \right\} \\ &\quad - \sum_{\nu=m+1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i) \\ &= k! \sum_{\nu=0}^{k} \left\{ f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{m-\nu}] - f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu}, x_0, \ldots, x_{m+1-\nu}] \right\} \\ &\quad \times \frac{D^{k-\nu}_{\gamma_i} \omega_{m+1-\nu}(x, \gamma_i)}{(k-\nu)!} + f_{\gamma_i}[x_0, \ldots, x_{m+1}] D^k_{\gamma_i} \omega_{m+1}(x, \gamma_i) - \sum_{\nu=m+1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i) \\ &= k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{m+1-\nu}] \frac{D^{k-\nu}_{\gamma_i} \omega_{m+1-\nu}(x, \gamma_i)}{(k-\nu)!} \\ &\quad \times (\gamma_i(x) - \gamma_i(x_{m+1-\nu})) - \sum_{\nu=m+2}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D^k_{\gamma_i} \omega_\nu(x, \gamma_i). \end{aligned}$$

The proof is completed. $\square$

As an immediate consequence of Theorem 1, the following corollary holds.

Corollary 1. *Let $D^{n+1}_{\gamma_i} f$ be continuous on $[a, b]$. For $0 \le k \le n$, we have*

$$\frac{D^k_{\gamma_i} R_n(x, \gamma_i)}{k!} = \sum_{\nu=0}^{k} \frac{f^{(n+1)}_{\gamma_i}(\xi_\nu)}{(n+1)!} \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu})), \tag{2.4}$$

where

$$\xi_\nu \in (\min\{x, x_0, \ldots, x_n\}, \max\{x, x_0, \ldots, x_n\}), \quad \nu = 0, 1 \ldots, k.$$

Moreover, assume that $M = \sup_{x\in[a,b]} |f^{(n+1)}_{\gamma_i}(x)|$, then

$$\left| D^k_{\gamma_i} R_n(x, \gamma_i) \right| \le \frac{Mk!}{(n+1)!} \sum_{\nu=0}^{k} \left| \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu})) \right|.$$

Proof. Taking $m = n$ in Theorem 1, we obtain

$$D^k_{\gamma_i} R_n(x, \gamma_i) = k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{n-\nu}] \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu})).$$

Since $D^{n+1}_{\gamma_i} f$ is continuous on $[a, b]$, using the fact that

$$f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{n-\nu}] = \frac{f^{n+1}_{\gamma_i}(\xi_\nu)}{(n+1)!},$$

where

$$\xi_\nu \in (\min\{x, x_0, \ldots, x_n\}, \max\{x, x_0, \ldots, x_n\}), \quad \nu = 0, 1 \ldots, k,$$

we obtain (2.4). Hence, this completes the proof. □

Before we give a remark on Corollary 1, a useful lemma is presented.

Lemma 1. *For $0 \le k \le n$, we have*

$$\frac{D^k_{\gamma_i} \omega_{n+1}(x, \gamma_i)}{k!} = \sum_{\nu=0}^{k} \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu})).$$

Proof. Replacing m by n in Eq. (2.3) and summing with respect to ν, the corollary is obtained. □

Remark 1. *In Corollary 1, formula (2.4) reduces to (2.1) when $k = 0$. As a special case, if all of $\frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x,\gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu}))$, $\nu = 0, 1, \ldots, k$ have the same sign, using Lemma 1 we obtain*

$$\frac{1}{k!} \left| D^k_{\gamma_i} \omega_{n+1}(x, \gamma_i) \right| = \sum_{\nu=0}^{k} \left| \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu})) \right|.$$

Then, we conclude that

$$\left| D^k_{\gamma_i} R_n(x, \gamma_i) \right| \le \frac{M}{(n+1)!} \left| D^k_{\gamma_i} \omega_{n+1}(x, \gamma_i) \right|.$$

In order to derive the next corollary, for $x, x + t \in [a, b]$ we define the modulus of continuity of $D^n_{\gamma_i} f$ by the equation

$$\omega(D^n_{\gamma_i} f, \delta) = \sup_{|t|\le\delta} |D^n_{\gamma_i} f(x + t) - D^n_{\gamma_i} f(x)|.$$

Corollary 2. *For $0 \le k \le n$, if $D^n_{\gamma_i} f$ is continuous on $[a, b]$, then*

$$\left| D^k_{\gamma_i} R_n(x, \gamma_i) \right| \le \frac{k!}{n!} \omega(D^n_{\gamma_i} f, h) \sum_{\nu=0}^{k} \left| \frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu-1}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu-1})) \right|,$$

where

$$h = \max\{x, x_0, \ldots, x_n\} - \min\{x, x_0, \ldots, x_n\}.$$

Proof. First, we put $m = n - 1$ in Theorem 1, then

$$D^k_{\gamma_i} R_n(x, \gamma_i) = k! \sum_{\nu=0}^{k} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{n-1-\nu}] \frac{D^{k-\nu}_{\gamma_i} \omega_{n-1-\nu}(x, \gamma_i)}{(k-\nu)!} \times (\gamma_i(x) - \gamma_i(x_{n-1-\nu})) - f_{\gamma_i}[x_0, \ldots, x_n] D^k_{\gamma_i} \omega_n(x, \gamma_i).$$

By Lemma 1 we have

$$\begin{aligned} D^k_{\gamma_i} R_n(x, \gamma_i) &= k! \sum_{\nu=0}^{k} \left(f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{n-1-\nu}] - f_{\gamma_i}[x_0, \ldots, x_n] \right) \\ &\quad \times \frac{D^{k-\nu}_{\gamma_i} \omega_{n-1-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-1-\nu})) \\ &= \frac{k!}{n!} \sum_{\nu=0}^{k} (f^{(n)}_{\gamma_i}(\xi^*_\nu) - f^{(n)}_{\gamma_i}(\xi^*)) \frac{D^{k-\nu}_{\gamma_i} \omega_{n-1-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-1-\nu})). \end{aligned}$$

Hence,

$$\begin{aligned} \left| D^k_{\gamma_i} R_n(x, \gamma_i) \right| &\leq \frac{k!}{n!} \sum_{\nu=0}^{k} \left| f^{(n)}_{\gamma_i}(\xi^*_\nu) - f^{(n)}_{\gamma_i}(\xi^*) \right| \left| \frac{D^{k-\nu}_{\gamma_i} \omega_{n-1-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-1-\nu})) \right| \\ &\leq \frac{k!}{n!} \omega(D^n_{\gamma_i} f, h) \sum_{\nu=0}^{k} \left| \frac{D^{k-\nu}_{\gamma_i} \omega_{n-1-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-1-\nu})) \right|. \quad \square \end{aligned}$$

Remark 2. *Similarly, when* $\frac{D^{k-\nu}_{\gamma_i} \omega_{n-\nu}(x, \gamma_i)}{(k-\nu)!} (\gamma_i(x) - \gamma_i(x_{n-\nu}))$, $\nu = 0, 1, \ldots, k$ *have the same sign, we also obtain*

$$\left| D^k_{\gamma_i} R_n(x, \gamma_i) \right| \leq \frac{1}{n!} \omega(D^n_{\gamma_i} f, h) \left| D^k_{\gamma_i} \omega_{n+1}(x, \gamma_i) \right|.$$

Lemma 2. *Suppose that* γ_i *is sufficiently smooth at all points* $x \in (a, b) \setminus \{y_i\}$. *Let* $D^k = \frac{d^k}{dx^k}$, $k = 1, 2, \ldots$, *then*

$$D^k = (\gamma_i'(x))^k D^k_{\gamma_i} + \sum_{j=1}^{k-1} c_{j,k}(x) D^j_{\gamma_i}, \tag{2.5}$$

where

$$c_{j,k}(x) = \begin{cases} (\gamma_i'(x))^k & j = k, \\ c'_{j,k-1}(x) + c_{j-1,k-1}(x)\gamma_i'(x), & j < k \text{ and } k \neq 1, \end{cases}$$
$$c_{0,q}(x) = 0, \quad q = 1, 2, \ldots$$

Proof. We proceed by induction on k. From the definition of D_{γ_i} it is clear that $D = \gamma_i'(x) D_{\gamma_i}$. Then Eq. (2.5) holds for $k = 1$. Now assume that (2.5) is valid for k. Therefore,

$$\begin{aligned} D^{k+1} &= D((\gamma_i'(x))^k D^k_{\gamma_i}) + \sum_{j=1}^{k-1} D(c_{j,k}(x) D^j_{\gamma_i}) \\ &= (\gamma_i'(x))^{k+1} D^{k+1}_{\gamma_i} + k!(\gamma_i'(x))^{k-1} \gamma_i''(x) D^k_{\gamma_i} + \sum_{j=1}^{k-1} (c'_{j,k}(x) D^j_{\gamma_i} + c_{j,k}(x)\gamma_i'(x) D^{j+1}_{\gamma_i}) \\ &= (\gamma_i'(x))^{k+1} D^{k+1}_{\gamma_i} + (k!(\gamma_i'(x))^{k-1} \gamma_i''(x) + c_{k-1,k}(x)\gamma_i'(x)) D^k_{\gamma_i} + \sum_{j=1}^{k-1} ((c'_{j,k}(x) + c_{j-1,k}(x)\gamma_i'(x)) D^j_{\gamma_i}). \end{aligned}$$

Due to $c'_{k,k}(x) = k!(\gamma_i'(x))^{k-1}\gamma_i''(x)$, it holds that

$$c_{j,k+1}(x) = c'_{j,k}(x) + c_{j-1,k}\gamma_i'(x), \quad j = 1, 2, \ldots, k,$$
$$c_{k+1,k+1} = (\gamma_i'(x))^{k+1}.$$

Thus, the lemma is valid for $k + 1$ and this completes the proof. $\square$

By Theorem 1 and Lemma 2, one readily obtains the following theorem.

Theorem 2. *Let γ_i be sufficiently smooth at all points $x \in (a, b) \setminus \{y_i\}$ and $f \in \mathcal{F}_{\gamma_i}^k[a, b]$. Then*

$$D^k R_n(x, \gamma_i) = (\gamma_i'(x))^k D_{\gamma_i}^k R_n(x, \gamma_i) + \sum_{j=1}^{k-1} c_{j,k}(x) D_{\gamma_i}^j R_n(x, \gamma_i),$$

where

$$c_{j,k}(x) = \begin{cases} (\gamma_i'(x))^k & j = k, \\ c_{j,k-1}'(x) + c_{j-1,k-1}(x)\gamma_i'(x) & j < k \text{ and } k \neq 1, \end{cases}$$

$$c_{0,q}(x) = 0, \quad q = 1, 2, \ldots,$$

and for $0 \le l \le k \le m$,

$$D_{\gamma_i}^l R_n(x, \gamma_i) = l! \sum_{\nu=0}^{l} f_{\gamma_i}[\underbrace{x, \ldots, x}_{\nu+1}, x_0, \ldots, x_{m-\nu}] \frac{D_{\gamma_i}^{l-\nu} \omega_{m-\nu}(x, \gamma_i)}{(l-\nu)!} (\gamma_i(x) - \gamma_i(x_{m-\nu})) - \sum_{\nu=m+1}^{n} f_{\gamma_i}[x_0, \ldots, x_\nu] D_{\gamma_i}^l \omega_\nu(x, \gamma_i).$$

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References

[1] S. Amat, S. Busquier, A. Escudero, J.C. Trillo, Lagrange interpolation for continuous piecewise smooth functions, J. Comput. Appl. Math. (2007) doi:10.1016/j. cam. 2007. 10. 011.
[2] I.R. Khan, R. Ohba, Closed-from expressions for the finite difference approximations of first and higher derivatives based on Taylor series, J. Comput. Appl. Math. 107 (1999) 179–193.
[3] I.R. Khan, R. Ohba, Digital differentiators based on Taylor series, IEICE Trans. Fund. E82-A (12) (1999) 2822–2824.
[4] I.R. Khan, R. Ohba, New finite difference formulas for numerical differentiation, J. Comput. Appl. Math. 126 (2001) 269–276.
[5] I.R. Khan, R. Ohba, Mathematical proof of explicit formulas for tap-coefficients of Taylor series based FIR digital differentiators, IEICE Trans. Fund. E84-A (6) (2001) 1581–1584.
[6] I.R. Khan, R. Ohba, Taylor series based finite difference approximation of high-degree derivatives, J. Comput. Appl. Math. 154 (2003) 115–124.
[7] I.R. Khan, R. Ohba, N. Hozumi, Mathematical proof of closed form expressions for finite difference approximations based on Taylor series, J. Comput. Appl. Math. 150 (2003) 303–309.
[8] X. Wang, The remainder of numerical differentiation formula (in Chinese), Hangzhou Daxue Xuebao 5 (1) (1978) 1–10. An announcement of the results appeared in Kexue Tong-bao, 24 (1979), 869–872.
[9] X. Wang, F. Cui, Stable Lagrangian numerical differentiation with the highest order of approximation, Sci. China. Ser. A 49 (2) (2006) 225–232.
[10] J.P. Li, General explicit difference formulas for numerical differentiation, J. Comput. Appl. Math. 183 (2005) 29–52.
[11] H. Wang, F. Cui, X. Wang, Explicit representations for local Lagrangian numerical differentiation, Acta. Math. Sinica 23 (2007) 365–372.